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On the algebraic connectivity of graphs as a function of genus

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Abstract

We find an upper bound on the algebraic connectivity of graphs of various genus. We begin by showing that for fixed $k \geqslant 1$, the graph of genus k of largest algebraic connectivity is a complete graph. We then find an upper bound for noncomplete graphs of a fixed genus $k \geqslant 1$ and we determine the values of k for which the upper bound can be attained. Finally, we find the upper bound of the algebraic connectivity of planar graphs (graphs of genus zero) and determine precisely which graphs attain this upper bound.
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1. Introduction and preliminaries

An **undirected graph** $\mathscr{G} = (V, E)$ on n vertices is a finite set V of cardinality n, whose elements are called **vertices**, together with a set E of two-element subsets of V called **edges**. It will be convenient to label the vertices $1, 2, \ldots, n$. We say that two vertices i and j are **adjacent** if there is an edge joining i and j. Furthermore, the **degree** of a vertex i, which we denote as d_i, is the number of edges incident to i.

With $\mathscr{G}$ we can associate the so-called **Laplacian matrix** which is the $n \times n$ matrix $L = (\ell_{i,j})$ whose entries are determined as follows:

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$$\ell_{i,j} = \begin{cases} -1, & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ d_i, & \text{if } i = j. \end{cases}$$

It is known that the Laplacian matrix is a symmetric positive semidefinite M-matrix.[2] We shall always consider its eigenvalues to have been arranged in a nondescending order: $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. For an extensive survey on the Laplacian matrix see Merris [7].

A graph $\mathscr{G}$ is called **connected** if there is a path linking any two of its vertices. Fiedler [3, p. 298] has shown that $\lambda_2 > 0$ if and only if $\mathscr{G}$ is connected and, partly for this reason, he has called $\mu(\mathscr{G}) := \lambda_2$ the **algebraic connectivity** of $\mathscr{G}$. Much work has been done concerning the algebraic connectivity of graphs. We refer the reader to [4,5].

We say that a graph can be **embedded** on a surface S if it can be drawn on S so that no two of its edges cross. Thus a graph is **planar** if it can be drawn on a plane so that no two edges cross. Let S_k denote a sphere with k handles. Thus S_0 is a sphere, S_1 is a torus, while S_2 is a double torus. We say that a graph $\mathscr{G}$ has **genus** k if $\mathscr{G}$ can be embedded on S_k but not on S_{k-1}. Observe that a graph is planar if and only if its genus is zero.

Intuitively, for all graphs on n vertices for some fixed n, graphs that have more edges are likely to be of higher genus and are also likely to have a larger algebraic connectivity. In Section 2, we focus on graphs of genus $k \geqslant 1$. We provide upper bounds for the algebraic connectivity of such graphs. We continue by showing that the upper bounds are obtained only by complete graphs. Thus it becomes natural to investigate upper bounds on the algebraic connectivity of noncomplete graphs of genus $k \geqslant 1$. We determine the values for k which the upper bound can be achieved while we also determine values of k for which our upper bound cannot be achieved. In Section 3, we focus on planar graphs and determine an upper bound on the algebraic connectivity for such graphs. We conclude by showing precisely which planar graphs attain this upper bound.

2. Graphs of genus $k \geqslant 1$

In this section, we will investigate the algebraic connectivity of graphs that can be embedded on surfaces S_k for all integers $k \geqslant 1$. We begin by introducing notation. Let $\nu(\mathscr{G})$ denote the vertex connectivity of $\mathscr{G}$, let $\delta(\mathscr{G})$ denote the degree of a vertex of $\mathscr{G}$ with the minimum degree, and let $\gamma(\mathscr{G})$ denote the genus of $\mathscr{G}$. We recall the following useful result from [3]:

Theorem 2.1. *If $\mathscr{G}$ is a noncomplete graph on n vertices, then*

$$\mu(\mathscr{G}) \leqslant \nu(\mathscr{G}) \leqslant \delta(\mathscr{G}) \leqslant n-1.$$

We should note that if $\mathscr{G} = K_n$, then $\mu(\mathscr{G}) = n$, $\delta(\mathscr{G}) = n-1$, and $\nu(\mathscr{G})$ is defined to be n. We now present a known consequence of the generalization of Euler's Identity (see [2, p. 247]).

Theorem 2.2. *If $\mathscr{G}$ is a connected graph on n vertices, m edges, and genus k, then*

$$k \geqslant \frac{m}{6} - \frac{n}{2} + 1.$$

Recall that for all graphs, the sum of the degrees of the vertices is $2m$. Thus for all graphs of genus k, Theorem 2.2 can be rewritten as $2m \leqslant 6n + 12(k-1)$. If $\mathscr{G}$ has n vertices, then since

[2] For more background material on nonnegative matrices and M-matrices see Berman and Plemmons [1].

the degree of each vertex is at least $\delta(\mathscr{G})$, it follows that $n\delta(\mathscr{G}) \leqslant 2m$. Hence $n\delta(\mathscr{G}) \leqslant 2m \leqslant 6n + 12(k-1)$. Therefore, for a graph $\mathscr{G}$ of genus k,

$$\delta(\mathscr{G}) \leqslant 6 + \left\lfloor \frac{12(k-1)}{n} \right\rfloor.$$

However, since $\delta(\mathscr{G}) \leqslant n-1$ for all graphs on n vertices, we have proven the following lemma:

Lemma 2.3. *Let $\mathscr{G}$ be a graph on n vertices that is of genus $k \geqslant 1$. Then*

$$\delta(\mathscr{G}) \leqslant \min\left\{n-1, 6 + \left\lfloor \frac{12(k-1)}{n} \right\rfloor\right\}.$$

Since $\mu(\mathscr{G}) \leqslant \delta(\mathscr{G})$ for any noncomplete graph $\mathscr{G}$, and since our goal is to find the best upper bound for $\mu(\mathscr{G})$ for a given genus $k \geqslant 1$, we want to find the positive integer value(s) n such that equality in Lemma 2.3 holds for each given $k \geqslant 1$. We do this in the following lemma. For the sake of simplicity, let

$$p_k := \left\lfloor \frac{7 + \sqrt{1+48k}}{2} \right\rfloor.$$

Lemma 2.4. *Let $\mathscr{G}$ be a graph of genus $k \geqslant 1$. Then $\delta(\mathscr{G}) \leqslant p_k - 1$.*

Proof. Observe that if

$$n - 1 = 6 + \left\lfloor \frac{12(k-1)}{n} \right\rfloor,$$

then

$$n = \left\lfloor \frac{7 + \sqrt{1+48k}}{2} \right\rfloor = p_k. \tag{2.1}$$

In such a case,

$$\delta(\mathscr{G}) \leqslant p_k - 1, \tag{2.2}$$

where $\mathscr{G}$ is of genus $k \geqslant 1$. Otherwise, since $n-1$ strictly increases in n while $6 + \lfloor 12(k-1)/n \rfloor$ monotonically decreases in n, the result follows from Lemma 2.3. □

We are now ready to prove the first main result of this section:

Theorem 2.5. *Let $\mathscr{G}$ be a graph of genus k where $k \geqslant 1$. If $\mathscr{G}$ is the complete graph on the largest number of vertices that can be embedded on S_k, then $\mu(\mathscr{G}) = p_k$. Otherwise, $\mu(\mathscr{G}) \leqslant p_k - 1$.*

Proof. By Lemma 2.4, if $\mathscr{G}$ is a graph of genus $k \geqslant 1$, then $\delta(\mathscr{G}) \leqslant p_k - 1$. By Theorem 2.1, $\mu(\mathscr{G}) \leqslant \delta(\mathscr{G})$ if $\mathscr{G}$ is not complete. Thus $\mu(\mathscr{G}) \leqslant p_k - 1$ for all noncomplete graphs of genus k. If $\mathscr{G}$ is complete then $\mu(\mathscr{G}) = p_k$ if and only if $\mathscr{G} = K_{p_k}$. It remains to show that K_{p_k} is the complete graph on the largest number of vertices that can be embedded on S_k. It is known from [9] that

$$\gamma(K_n) = \left\lceil \frac{(n-4)(n-3)}{12} \right\rceil. \tag{2.3}$$

Letting $n = p_k$, (2.3) shows that $\gamma(K_{p_k}) = k$. Thus K_{p_k} is the complete graph on the largest number of vertices that can be embedded on S_k. This proves the theorem. $\square$

An interesting question now arises. Given a surface S of genus k, does there necessarily exist a noncomplete graph $\mathscr{G}$ of genus k such that $\mu(\mathscr{G}) = p_k - 1$? To begin answering this question, we prove the following lemma:

Lemma 2.6. *If $k \geqslant 1$ is an integer such that $p_k - 7 = 12(k-1)/p_k$, then $\mu(\mathscr{G}) < p_k - 1$ for all noncomplete graphs $\mathscr{G}$ of genus k.*

Proof. Let $\mathscr{G}$ be a noncomplete graph on n vertices of genus $k \geqslant 1$ where k is an integer such that $p_k - 7 = 12(k-1)/p_k$. Lemma 2.4 dictates that $\delta(\mathscr{G}) \leqslant p_k - 1$. Suppose $\delta(\mathscr{G}) = p_k - 1$. This implies that $n \geqslant p_k$. But since, $\mathscr{G}$ is noncomplete, $n \geqslant p_k + 1$. Thus

$$p_k - 1 = 6 + \frac{12(k-1)}{p_k} = 6 + \left\lfloor \frac{12(k-1)}{p_k} \right\rfloor > 6 + \left\lfloor \frac{12(k-1)}{n} \right\rfloor,$$

where the second equality follows from the fact that $12(k-1)/p_k$ is an integer. So by Lemma 2.3, if $n \geqslant p_k + 1$, then

$$\delta(\mathscr{G}) \leqslant 6 + \left\lfloor \frac{12(k-1)}{n} \right\rfloor < 6 + \left\lfloor \frac{12(k-1)}{p_k} \right\rfloor = p_k - 1. \tag{2.4}$$

This contradicts $\delta(\mathscr{G}) = p_k - 1$. Hence $\delta(\mathscr{G}) < p_k - 1$, and thus it follows that $\delta(\mathscr{G}) \leqslant p_k - 2$. Therefore, by Theorem 2.1, $\mu(\mathscr{G}) \leqslant p_k - 2$. Thus there does not exist a noncomplete graph $\mathscr{G}$ of genus $k \geqslant 1$ such that $\mu(\mathscr{G}) = p_k - 1$ if k is an integer such that $p_k - 7 = 12(k-1)/p_k$. $\square$

Remark. If k is an integer such that $p_k - 7 \neq 12(k-1)/p_k$, then $12(k-1)/p_k$ need not be an integer. Hence in this case, if $n \geqslant p_k + 1$, it would be possible for $\lfloor 12(k-1)/p_k \rfloor$ to equal $\lfloor 12(k-1)/n \rfloor$ in (2.4).

It now behooves us to determine which integers $k \geqslant 1$ are such that $p_k - 7 = 12(k-1)/p_k$. We determine this in the following claim:

Claim 2.7. *$p_k - 7 = 12(k-1)/p_k$ if and only if $\sqrt{1+48k}$ is an integer.*

Proof. Observe that $1 + 48k$ is an odd integer for all integers $k \geqslant 1$. Thus if $\sqrt{1+48k}$ is an integer, it must also be odd. Hence $(7 + \sqrt{1+48k})/2$ is an integer if and only if $\sqrt{1+48k}$ is an (odd) integer. Therefore, this claim is proven if we can show $(7 + \sqrt{1+48k})/2$ is an integer if and only if $p_k - 7 = 12(k-1)/p_k$.

Suppose $(7 + \sqrt{1+48k})/2$ is an integer. Then $p_k = (7 + \sqrt{1+48k})/2$. Thus, rewriting this expression yields $p_k - 7 = 12(k-1)/p_k$.

Now suppose $(7 + \sqrt{1+48k})/2$ is not an integer. Then $p_k < (7 + \sqrt{1+48k})/2$. Rewriting this expression will yield $p_k - 7 < 12(k-1)/p_k$. $\square$

We are now ready to prove the second main result of this section which shows values for k in which $\mu(\mathscr{G}) < p_k - 1$ for all graphs of genus k.

Theorem 2.8. *If any of the following hold for some positive integer c:*

(a) $k = c(12c - 1)$,
(b) $k = c(12c + 1)$,
(c) $k = (4c - 1)(3c - 1)$,
(d) $k = (4c + 1)(3c + 1)$,

then $\mu(\mathscr{G}) < p_k - 1$ for all noncomplete graphs $\mathscr{G}$ of genus k.

Proof. From Claim 2.7, we see that $p_k - 7 = 12(k-1)/p_k$ if and only if $\sqrt{1+48k}$ is an integer. But $\sqrt{1+48k}$ is an integer if and only if $1+48k$ is a perfect square, i.e. $1+48k = x^2$ for some integer x. This is equivalent to $(x+1)(x-1) = 48k$ for some integer x. Letting $y = x+1$, we see that $p_k - 7 = 12(k-1)/p_k$ if and only if there exists an integer y such that $y(y-2)$ is divisible by 48. In such a case,

$$k = \frac{y(y-2)}{48}. \tag{2.5}$$

Observe that $y(y-2)$ is divisible by 48 if and only if any of the following hold:

(i) $y = 24c$ for some positive integer c,
(ii) $y = 24c + 2$ for some positive integer c,
(iii) $y = 24c - 6$ for some positive integer c,
(iv) $y = 24c + 8$ for some positive integer c.

Plugging (i)–(iv) into (2.5) yields (a)–(d), respectively. This together with Lemma 2.6 proves the theorem. □

The first nine values for k in which $\mu(\mathscr{G}) < p_k - 1$ for all noncomplete graphs of genus k that Theorem 2.8 produces are 6, 11, 13, 20, 35, 46, 50, 63, 88. Theorem 2.8 leads us to two questions. First, are there values for k in which there exists a noncomplete graph $\mathscr{G}$ such that $\mu(\mathscr{G}) = p_k - 1$? The answer is the affirmative. For example $\gamma(K_{3,3,3}) = 1$ (see [10]), and observe that $\mu(K_{3,3,3}) = p_1 - 1 = 6$. Similarly, $\gamma(K_{4,4,4}) = 3$ and note that $\mu(K_{4,4,4}) = p_3 - 1 = 8$.

The second question worth investigating is if there are other values for k, other than those listed in Theorem 2.8, in which $\mu(\mathscr{G}) < p_k - 1$ for all noncomplete graphs $\mathscr{G}$ of genus k. Since finding the genus of a graph can be very difficult, this remains an open question.

It should also be noted that for any graph $\mathscr{G}$, we can append a path to any vertex without changing the genus of $\mathscr{G}$. Let $\mathscr{G}_{v,n}$ be the graph obtained from $\mathscr{G}$ by appending a path of on n vertices to a fixed vertex v of $\mathscr{G}$. Since

$$\lim_{n\to\infty} \mu(\mathscr{G}_{v,n}) = 0$$

for any graph $\mathscr{G}$ and any vertex $v \in \mathscr{G}$, it follows for any $k \geqslant 0$ that the lower bound on the algebraic connectivity of graphs on genus k is zero.

3. Planar graphs

In the previous section, we used the minimum degree of a vertex in a graph in order to aid us in determining upper bounds on the algebraic connectivity of a graph of genus $k \geqslant 1$. We will attempt to use the same ideas for planar graphs. We first note the following well known theorem (see [2]):

Theorem 3.1. *If $\mathscr{G}$ is a planar graph, then there must exist a vertex in $\mathscr{G}$ of degree* 5 *or less.*

Recall that if $\mathscr{G}$ is of genus $k \geqslant 1$, then $\delta(\mathscr{G}) \leqslant p_k - 1$ and the complete graph K_{p_k} can be embedded onto S_k. However, if $\mathscr{G}$ is planar then $\delta(\mathscr{G}) \leqslant 5$ but K_6 is not planar. In other words, for surfaces S_k where $k \geqslant 0$, the surface S_0 is the only surface in which we cannot embed the complete graph whose vertices each have the largest possible minimum degree. Therefore, the methods for determining the upper bound on the algebraic connectivity for graphs of genus $k \geqslant 1$ will not work in the case that $k = 0$. In fact, K_5 is not planar either. Thus we will have to use alternate methods to those used in Section 2. The aim of this section is to prove the following theorem:

Theorem 3.2. *If $\mathscr{G}$ is a planar graph, then $\mu(\mathscr{G}) \leqslant 4$. Moreover, equality holds if and only if $\mathscr{G} \cong K_4$ or $\mathscr{G} \cong K_{2,2,2}$.*

First we note that the complete graph on the largest number of vertices that is planar is K_4 and that $\mu(K_4) = 4$. Hence by Theorem 2.1, we need only consider noncomplete planar graphs such that $\delta(\mathscr{G}) = 4$ and $\delta(\mathscr{G}) = 5$. We will prove this as two claims. Claim 3.4 will concern the case where $\delta(\mathscr{G}) = 4$, while Claim 3.11 will concern the case where $\delta(\mathscr{G}) = 5$. We will first begin with Claim 3.4. To prove this claim, we need the following theorem from [6]. Note that the **join** of two graphs $\mathscr{G}_1$ and $\mathscr{G}_2$, written $\mathscr{G}_1 \vee \mathscr{G}_2$, is the graph created from $\mathscr{G}_1 \cup \mathscr{G}_2$ by joining every vertex of $\mathscr{G}_1$ to every vertex of $\mathscr{G}_2$.

Theorem 3.3. *Let $\mathscr{G}$ be a noncomplete, connected graph on n vertices. Then $\nu(\mathscr{G}) = \mu(\mathscr{G})$ if and only if $\mathscr{G}$ can be written as $\mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - \nu(\mathscr{G})$ vertices and $\mathscr{G}_2$ is a graph on $\nu(\mathscr{G})$ vertices with $\mu(\mathscr{G}_2) \geqslant 2\nu(\mathscr{G}) - n$.*

We now begin with the case where $\delta(\mathscr{G}) = 4$:

Claim 3.4. *Let $\mathscr{G}$ be a noncomplete planar graph on n vertices such that $\delta(\mathscr{G}) = 4$, then $\mu(\mathscr{G}) \leqslant 4$ and equality holds if and only if $\mathscr{G} \cong K_{2,2,2}$.*

Proof. Suppose $\mathscr{G}$ is a planar graph such that $\delta(\mathscr{G}) = 4$. Then by Theorem 2.1, $\mu(\mathscr{G}) \leqslant 4$. If $\mu(\mathscr{G}) = \nu(\mathscr{G}) = 4$, then by Theorem 3.3, $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$ where $\mathscr{G}_1$ is a disconnected graph $n - 4$ vertices, while $\mathscr{G}_2$ is a graph on four vertices with $\mu(\mathscr{G}_2) \geqslant n - 8$. Since $\mathscr{G}$ is planar, by Kuratowski's Theorem (see [2, p. 235]), $\mathscr{G}$ cannot contain $K_{3,3}$ as a subgraph. Therefore, since $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$ and since $\mathscr{G}_2$ has four vertices, it follows that $\mathscr{G}_1$ has either one vertex or two vertices. If $\mathscr{G}_1$ has one vertex, then $\mathscr{G}_1 = K_1$. But, $\mu(K_1 \vee \mathscr{G}_2) < 4$ for all graphs $\mathscr{G}_2$ on four vertices in which $K_1 \vee \mathscr{G}_2$ is planar. If $\mathscr{G}_1$ has two vertices, then since $\mathscr{G}_1$ must be disconnected, it follows that $\mathscr{G}_1 = E_2$, where E_2 is the empty graph on two vertices. However, $\mu(E_2 \vee \mathscr{G}_2) \leqslant 4$ for all graphs $\mathscr{G}_2$ on four vertices in which $E_2 \vee \mathscr{G}_2$ is planar, and equality holds if and only if $\mathscr{G}_2 = C_4$ where C_4 is the cycle on four vertices. But $E_2 \vee C_4 = K_{2,2,2}$ (which is planar). Since $\mu(K_{2,2,2}) = 4$, the claim is proven. $\square$

To prove Theorem 3.2, we need to show that $\mu(\mathscr{G}) < 4$ for all planar graphs $\mathscr{G}$ in which $\delta(\mathscr{G}) = 5$. For this case, we use the concept of the **isoperimetric number** to determine an upper bound on the algebraic connectivity of such graphs. Let $\mathscr{G}$ be a graph on n vertices and let X be a subset of the set V of vertices of $\mathscr{G}$. Let E_X be the set of edges e where e is incident to exactly one vertex in X. Let $|X|$ denote the cardinality of a set X. Then the isoperimetric number of $\mathscr{G}$ is

$$i(\mathscr{G}) := \min \frac{|E_X|}{|X|},$$

where the minimum is taken over all subsets X of V where $1 \leqslant |X| \leqslant n/2$. The following theorem is known from [8]:

Theorem 3.5. *If $\mathscr{G}$ is a graph on $n > 3$ vertices, then $\mu(\mathscr{G}) \leqslant 2i(\mathscr{G})$.*

For our purposes, since the isoperimetric number is a minimum over all appropriate subsets X of V, we need only the following (weaker) corollary:

Corollary 3.6. *Let $\mathscr{G}$ be a graph on $n > 3$ vertices and let X be any subset of V such that $1 \leqslant |X| \leqslant n/2$. Then $\mu(\mathscr{G}) \leqslant 2|E_X|/|X|$.*

Before we are able to use the concept of the isoperimetric number, we need two lemmas concerning the structure of planar graphs. Lemma 3.8 deals with how a graph is drawn in the plane. Note that if a planar graph is drawn so that no edges cross (by definition, this is always possible for planar graphs), then the graph divides the plane into **regions**. A graph is outerplanar if it can be drawn such that the vertices lie on a cycle and all edges can be drawn passing through the interior region of the cycle so that no two edges cross. The following theorem concerning outerplanar graphs is well known (see [11, p. 240]):

Theorem 3.7. *If $\mathscr{G}$ is outerplanar, then there exists at least two nonadjacent vertices of degree two.*

We now prove the following lemma:

Lemma 3.8. *Let $\mathscr{G}$ be a planar graph drawn in the plane. Let R be a region bounded by a set of vertices forming a cycle C such that either*

(a) *for each vertex $v \in C$ we have $2 \leqslant \deg(v) \leqslant 4$; or*
(b) *there exists exactly two adjacent vertices $w, z \in C$ such that $\deg(w) \geqslant \deg(z) \geqslant 5$, while for all remaining vertices $v \in C$ we have $2 \leqslant \deg(v) \leqslant 4$.*

Then it is impossible to create a graph $\mathscr{H}$ from $\mathscr{G}$ by adding edges joining nonadjacent vertices of C, each of which passes through R, so that $\mathscr{H}$ is planar, and all vertices in C have degree five or greater in $\mathscr{H}$.

Proof. Let $\mathscr{G}_c$ be the graph formed by C and the edges being added that pass through R. Observe that $\mathscr{G}_c$ is outerplanar. Thus by Theorem 3.7 there exists at least two nonadjacent vertices in $\mathscr{H}$ with degree two. If (a) holds then such vertices will have degree at most four in $\mathscr{G}$. If (b) holds then since w and z are adjacent, it follows from Theorem 3.7 that at least one vertex in C other than w or z will have degree two in $\mathscr{G}_c$. Thus at least one vertex in C will have degree at most four in $\mathscr{G}$. □

There are some planar graphs in which adding an edge between any pair of nonadjacent vertices would result in a nonplanar graph. Such a graph is called **maximal planar**. Observe that if a maximal planar graph is drawn so that no edges cross, then every region created by the graph is bounded by three edges. Thus we say that every region of a maximal planar graph is a **triangle** and that each vertex is the center of a **wheel subgraph**. A wheel is a graph created by joining a central vertex to every vertex of a cycle.

We use Lemma 3.8 to prove the next important lemma which deals explicitly with graphs $\mathscr{G}$ where $\delta(\mathscr{G}) = 5$ and $\mathscr{G}$ is maximal planar. In this lemma, let $d(v, w)$ denote the distance between vertices v and w and let $\text{Adj}(v) := \{x | d(x, v) \leqslant 1\}$.

Lemma 3.9. *Let $\mathscr{G}$ be a maximal planar graph on n vertices such that $\delta(\mathscr{G}) = 5$. Then $|\text{Adj}(v)| \leqslant \lceil n/2 \rceil$ for all $v \in \mathscr{G}$. Moreover, for each vertex $v \in \mathscr{G}$, there exists a vertex $w \in \mathscr{G}$ such that $d(v, w) \geqslant 3$.*

Proof. Consider $v \in \mathscr{G}$ and let $\deg v = k$. We need to show that $k + 1 \leqslant \lceil n/2 \rceil$. Since $\mathscr{G}$ is maximal planar, v is the center of some wheel. Therefore let v be adjacent to $v_1, \ldots, v_k$ where v_i is adjacent to v_{i+1} for all $1 = 1, \ldots, k - 1$ and where v_k is adjacent to v_1. Let Z be the set of vertices outside $\text{Adj}(v)$ that are adjacent to at least one of $v_1, \ldots, v_k$. Since each edge in a maximal planar graph lies on the boundary of two triangle regions, and since each vertex has degree at least five, it follows that for each $i = 1, \ldots, k - 1$, there exists a unique vertex $w_i \in Z$ such that w_i is adjacent to both v_i and v_{i+1} (otherwise there would be a vertex of degree four). Similarly, there exists a vertex $w_k \in Z$ adjacent to both v_k and v_1. There may also be vertices in Z adjacent to exactly one vertex in $\text{Adj}(v)$. Hence $|Z| \geqslant k$. Let $\hat{\mathscr{G}}$ be the subgraph of $\mathscr{G}$ that is depicted in Fig. 1 (where $k = 6$ and vertices v_1, v_2, v_3, and v_5 have other vertices in Z adjacent to them). The vertices of $\hat{\mathscr{G}}$ are precisely the vertices in $\text{Adj}(v) \cup Z$. (Note that we cannot conclude that $\hat{\mathscr{G}}$ is the subgraph induced by the vertices in $\text{Adj}(v) \cup Z$ because there may be additional edges joining pairs of these vertices.) Therefore, $|\hat{\mathscr{G}}| \geqslant 2k + 1$. The theorem is proven if there exists a vertex in $\mathscr{G}$ that is not in $\hat{\mathscr{G}}$ for that would imply $n \geqslant 2k + 2$.

We first must show that the vertices in Z induce a cycle. Let us suppose not and deduce a contradiction. If the vertices of Z do not induce a cycle, then there are two cases to consider:

Case (i): There exists two vertices in $\text{Adj}(v)$ that are adjacent in $\mathscr{G}$ but the edge joining them is not an edge in $\hat{\mathscr{G}}$.

Case (ii): There exists two nonadjacent vertices in $\text{Adj}(v)$ that are adjacent to a common vertex in $x \in Z$.

First we consider Case (i). Let v_p and v_q where $p < q$ be vertices in $\text{Adj}(v)$ satisfying the statement of Case (i) such that none of the vertices v_i where $p + 1 \leqslant i \leqslant q - 1$ satisfy Case (i) or Case (ii). Let Z' be the set of vertices in Z that are adjacent to such vertices v_i. Then since $\mathscr{G}$ is maximal planar, the vertices in Z' form a path P in $\mathscr{G}$. Let $\mathscr{H}$ be the subgraph of $\mathscr{G}$ formed from $\hat{\mathscr{G}}$ by adding the edges of P (see Fig. 2). Observe that in $\mathscr{H}$, the vertex v_p followed by the vertices in Z' followed by v_q followed by v_p forms a cycle C. Let R denote the region that is the interior of C.

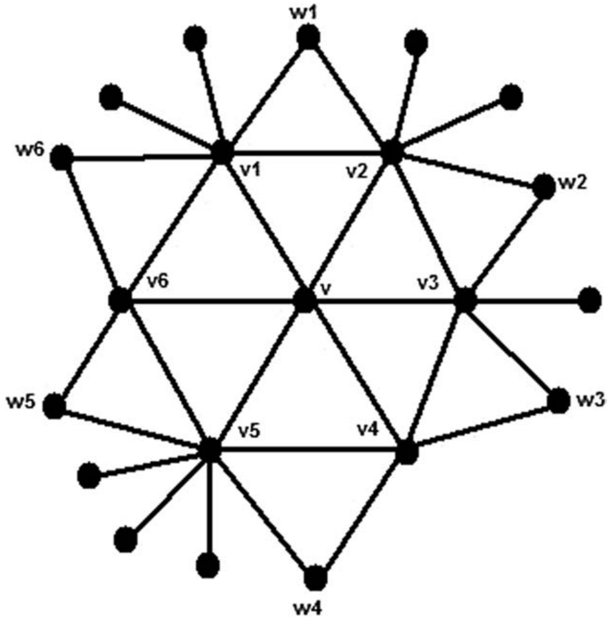

Fig. 1.

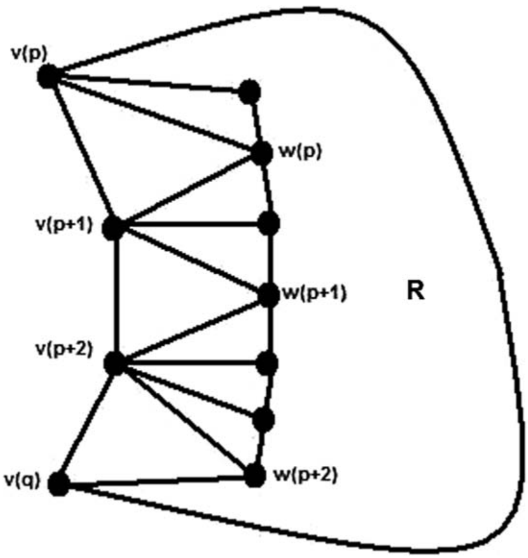

Fig. 2.

Thus to create $\mathscr{G}$ from $\mathscr{H}$, edges would need to be added which pass through R. Observe that in $\mathscr{H}$, that $2 \leqslant \deg(z) \leqslant 4$ for all $z \in Z'$. Thus the only two vertices in C which can possibly have a degree at least five in $\mathscr{G}$ are v_p and v_q which are adjacent in C. Thus by Lemma 3.8(b),

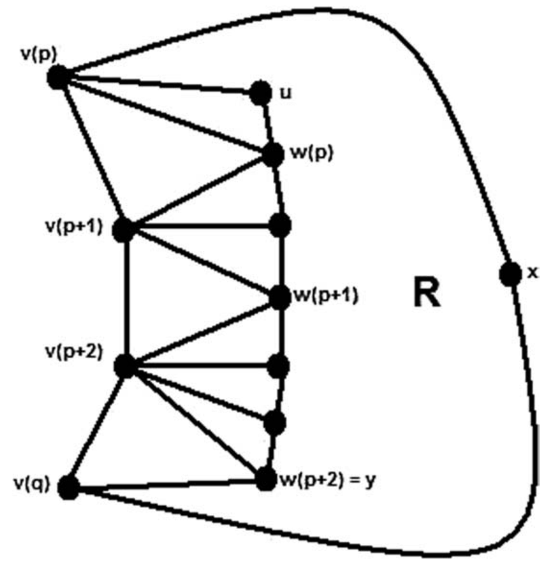

Fig. 3.

it is impossible to add edges passing through R to create $\mathscr{G}$ such that all vertices in C will have degree at least five in $\mathscr{G}$. Hence Case (i) cannot hold.

Now we consider Case (ii). Let v_p and v_q where $p < q$ be vertices in $\mathrm{Adj}(v)$ satisfying the statement of Case (ii) such that none of the vertices v_i where $p + 1 \leqslant i \leqslant q - 1$ satisfy Case (i) or Case (ii). Let Z' be as in Case (i) and recall that the vertices in Z' form a path P in $\mathscr{G}$. Let $\mathscr{H}$ be the subgraph of $\mathscr{G}$ formed from $\hat{\mathscr{G}}$ by adding the edges of P. Observe that in $\mathscr{H}$, the vertex v_p followed by the vertices in Z' followed by v_q followed by x followed by v_p forms a cycle C. Let R be the interior of C and let u and y be the vertices besides x that are adjacent to v_p and v_q, respectively in C (Fig. 3). Thus to create $\mathscr{G}$ from $\mathscr{H}$, edges would need to be added which pass through R. Observe that in $\mathscr{H}$, that $2 \leqslant \deg(z) \leqslant 4$ for all $z \in Z'$. Also note in $\mathscr{H}$ that $\deg(x) = 2$, and $\deg(u), \deg(y) \leqslant 3$. Thus the only two vertices in C which can possibly have a degree of at least five in $\mathscr{G}$ are v_p and v_q. We now have three subcases to consider.

Subcase (a): Vertices v_p and v_q are adjacent in $\mathscr{G}$. Then the vertex v_p followed by the vertices in P followed by v_q and then v_p forms a cycle in $\mathscr{G}$. Let R' be the region bounded by this cycle. Hence by the reasoning in Case (i), edges passing though R' cannot be added so that every vertex in the resulting graph has degree five or greater.

Subcase (b): Vertex v_p or v_q, say v_p, is adjacent to a vertex $s \in Z'$. Then R is divided into two regions. Consider the region that is not bounded by x and observe that v_p and s are the only vertices whose degree can possibly be at least five. Thus by the reasoning of Case (i), adding edges passing through this region cannot result in a graph in which every vertex is of degree five or greater.

Subcase (c): Neither vertices v_p nor v_q are adjacent to other vertices in Z' (except for those on C). Then since $\mathscr{G}$ is maximal planar and thus every region in $\mathscr{G}$ is a triangle, x is adjacent to both u and y in $\mathscr{G}$. Since $\deg(u), \deg(y) \leqslant 3$ in $\mathscr{H}$, adding such edges increases the degrees of u and y to at most four. Thus we have created a cycle in which each vertex has degree at most four. Thus by Lemma 3.8(a), we cannot continue to add edges such that the resulting graph has all vertices of degree five or greater.

Subcases (a) through (c) show that Case (ii) cannot hold. Therefore in $\mathscr{G}$, the vertices of Z induce a cycle where $3 \leqslant \deg(z) \leqslant 4$ for all vertices $z \in Z$. Suppose that all vertices of $\mathscr{G}$ are also vertices of $\hat{\mathscr{G}}$. Since all the vertices of $\mathscr{G}$ have minimum degree five and are vertices in $\hat{\mathscr{G}}$, in order to create $\mathscr{G}$ from $\hat{\mathscr{G}}$, edges joining nonadjacent vertices of Z would have to be added. By the construction of $\hat{\mathscr{G}}$, all such edges would have to pass through the same region. By Lemma 3.8(a), it is impossible to add such edges so that the resulting graph would be both planar and have minimum degree five. Therefore there must exist a vertex in $\mathscr{G}$ that is not a vertex in $\hat{\mathscr{G}}$. Hence $n \geqslant 2k+2$ and therefore $k+1 \leqslant \lceil n/2 \rceil$. Since $|\mathrm{Adj}(v)| \leqslant \lceil n/2 \rceil$ for all vertices $v \in \mathscr{G}$, then for each vertex v, there exists a vertex $w \notin \mathrm{Adj}(v) \cup Z$. Thus $d(v, w) \geqslant 3$. $\square$

If $\mathscr{G}$ is a maximal planar graph on $n \geqslant 4$ vertices, then every vertex of $\mathscr{G}$ is the center of a wheel subgraph. Thus Lemma 3.9 says that if $\mathscr{G}$ is a maximal planar graph such that $\delta(\mathscr{G}) = 5$, then there exists two vertices w_1 and w_2 that are the centers of wheels W_1 and W_2, respectively, where W_1 and W_2 do not share any common vertices or edges and such that one wheel consists of no more than $\lceil n/2 \rceil$ vertices while the other wheel consists of no more than $\lfloor n/2 \rfloor$ vertices. Observe that the vertices adjacent to w_1 and w_2 form disjoint cycles, C_1 and C_2, respectively.

Suppose $|W_1 \cup W_2| < n$. Then there exists a vertex $v \notin W_1 \cup W_2$ that is adjacent to vertices in C_1 or C_2. Since $\deg(v) \geqslant 5$, it follows that v is adjacent to at least two vertices of C_1 or two vertices of C_2. Without loss of generality, let v be adjacent to at least two vertices of C_1, say v_1 and v_2. Let v_0 and v_3 be the vertices on C_1 not adjacent to v such that v_1 is adjacent to v_0, and v_2 is adjacent to v_3. Hence we can extend C_1 to form the cycle $\hat{C}_1$ by beginning at v_0, then travelling to v_1, v, v_2, v_3 and then travelling the remaining vertices of C_1 returning to v_0 (Fig. 4).

We can continue extending $\hat{C}_1$ and $\hat{C}_2$ in this manner until we have two disjoint sets X and Y such that $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$, and the vertices in each set that are adjacent to the vertices in the other set each form a cycle. We state this more formally as follows:

Observation 3.10. If $\mathscr{G}$ is a maximal planar graph on n vertices in which $\delta(\mathscr{G}) = 5$, then there exists disjoint sets of vertices X and Y in which $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$ such that all of the following hold:

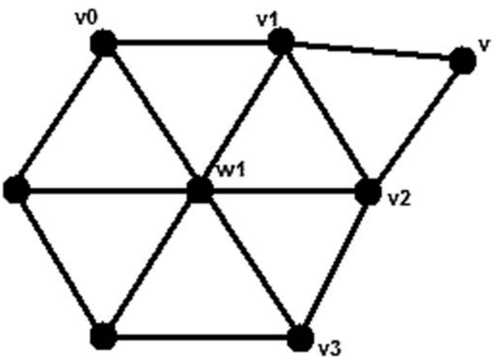

Fig. 4.

(a) The vertices in X that are adjacent to vertices in Y form a cycle.
(b) The vertices in Y that are adjacent to vertices in X form a cycle.
(c) There exists a vertex $x \in X$ that is not adjacent to any vertex in Y.
(d) There exists a vertex $y \in Y$ that is not adjacent to any vertex in X.

We can now prove the case where $\delta(\mathscr{G}) = 5$. Before doing so, we introduce notation in order to make the proof more transparent. In parts (a) and (b), of Observation 3.10, we will let X_c and Y_c denote the respective cycles formed. We let $x_1, \ldots, x_k$ be the vertices of X_c, and let $y_1, \ldots, y_m$ be the vertices of Y_c. We will orient each cycle so that x_i is the ith vertex of X_c, and y_i is the ith vertex of Y_c. This allows us to use the idea of vertices being *consecutive*. Thus, in $X_c(Y_c)$, we regard $x_{i+1}(y_{i+1})$ as the vertex that directly follows $x_i(y_i)$ for $i = 1, \ldots, k-1(m-1)$, and $x_1(y_1)$ as the vertex that directly follows $x_k(y_m)$. Hence in a set of consecutive vertices of either X_c or Y_c, the words *first* and *last* have meaning. We now proceed with the proof:

Claim 3.11. *Let $\mathscr{G}$ be a maximal planar graph on n vertices in which $\delta(\mathscr{G}) = 5$. Then $\mu(\mathscr{G}) < 4$.*

Proof. Let X and Y be sets of vertices in $\mathscr{G}$ that are in accordance with Observation 3.10. Let $X_c = x_1, x_2, \ldots, x_k, x_1$ be the (cycle of) vertices of X that are adjacent to vertices in Y and let $Y_c = y_1, y_2, \ldots, y_m, y_1$ be the (cycle of) vertices of Y that are adjacent to vertices in X. By Observation 3.10, $k + m \leqslant n - 2$. We will show that there are at most $n - 2$ edges joining vertices in X with vertices in Y.

Since $\mathscr{G}$ is maximal planar, every region is a triangle. Therefore, each $x_i \in X_c$ is adjacent to consecutive vertices (possibly only one vertex) in Y_c. Similarly, each $y_i \in Y_c$ is adjacent to consecutive vertices (possibly only one vertex) in X_c. We will now label the edges joining vertices in X_c to vertices in Y_c in correspondence with the vertices in X_c and Y_c; we will show that a one-to-one correspondence exists. Since each vertex in X_c is adjacent to consecutive vertices in Y_c, each vertex x_i has a last vertex in Y_c to which it is adjacent. Label the edge joining x_i to such a vertex as x_i (see Fig. 5). (If a vertex x_i is adjacent to all vertices in Y_c, then label the edge joining x_i to y_m as x_i.) Similarly, considering the vertices in Y_c, since each is adjacent to consecutive

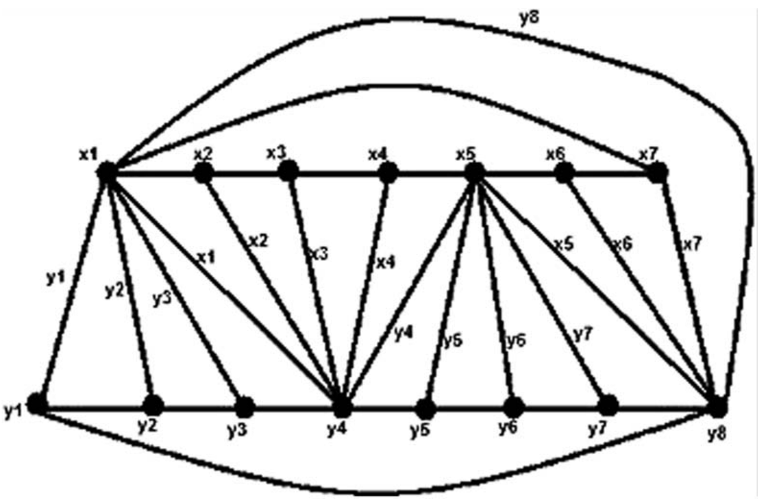

Fig. 5.

vertices in X_c, each vertex y_i has a last vertex in X_c to which it is adjacent. Label the edge joining the y_i to such a vertex as y_i. (If a vertex y_i is adjacent to all vertices in X_c, then label the edge joining y_i to x_k as y_i.) Thus we have created a one-to-one correspondence between the vertices in $X_c \cup Y_c$ and the edges joining the vertices in X_c to vertices in Y_c. Hence $|E_X| = k + m \leqslant n - 2$. Hence by Corollary 3.6.

$$\mu(\mathscr{G}) \leqslant \frac{2|E_X|}{|X|} \leqslant \frac{2(n-2)}{\lfloor n/2 \rfloor} \leqslant \frac{2(n-2)}{(n-1)/2} = \frac{4(n-2)}{n-1} < 4. \qquad \square$$

Claims 3.4 and 3.11 show that if $\mathscr{G}$ is a noncomplete maximal planar graph, then $\mu(\mathscr{G}) \leqslant 4$ where equality holds if and only if $\mathscr{G} = K_{2,2,2}$. Note that every planar graph $\mathscr{P}$ is the subgraph of some maximal planar graph $\mathscr{G}$. Since $\mathscr{P}$ is a subgraph of $\mathscr{G}$, it follows that $\mu(\mathscr{P}) \leqslant \mu(\mathscr{G})$ (see [3]). Finally, since no subgraph $\mathscr{P}$ of $K_{2,2,2}$ is such that $\mu(\mathscr{P}) = 4$, Theorem 3.2 is proven.

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